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PROGRAM CONSTRUCTIONS FOR POSITION GAME CONTROL

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The paper is a continuation of [1]. The method of auxiliary program constructions [2] is developed here for linear evolution systems.

1. We consider the evolution system

$$y^{\cdot} = A(t)y + f(t, u, v), \ u \in P, \ v \in Q \tag{1.1}$$

in a Hilbert space $\{y\} = Y$. We restrict ourselves to the cases when the solutions $y[t] = y[t, t_*, y_*, u[\cdot], v[\cdot]], \ t \geqslant t_*$, can be treated somehow or other proceeding from the weak Cauchy equality

$$\langle q \cdot y[t] \rangle = \langle q \cdot T(t, t_*) y_* \rangle + \int_{t_*}^{t} \langle q \cdot T(t, \tau) f(\tau, u[\tau], v[\tau]) \rangle d\tau \tag{1.2}$$

Here $T(t, \tau), \ t \geqslant \tau$, is a suitable semigroup operator, q is an arbitrary element of Y and $\langle q \cdot y \rangle$ is the scalar product. The norm of y is denoted by the symbol $\| y \|$ Let $y_\vartheta[t]$ be the value obtained from $y[t]$ in (1.2) by the transformation $T(\vartheta, t)$. Let A_γ be some linear operator from Y into some Hilbert space $Y^{(\gamma)}$ and let $y^{(\gamma)}[t]$ be the value obtained from $y_\vartheta[t]$ by transformation A_γ. We assume that for the evolution system [1]

$$y^{(\gamma)}[\cdot] = \{ y^{(\gamma)}[t], \ t_* \leqslant t \leqslant t^* \} \in \{ y^{(\gamma)}[\cdot] \mid y^{(\gamma)}[t_*], \ \Pi[t_*, t^*) \} \tag{1.3}$$

induced by (1.2) and by the transformation $A_\gamma T(\vartheta, t)$ we can construct a w-model which approximates (1.3) from below [1] and is described by the unified differential equation [3]

$$w^{\cdot} = q\xi(t, q) + p, \quad \| q \| = 1, \quad p \in P(q) \tag{1.4}$$

where $P(q)$ is a convex weak compactum in $Y^{(\gamma)}$ such that $\langle p \cdot q \rangle \geqslant 0$ for $p \in P(q)$ and in $P(q)$ there is an element p_* for which $\langle p_* \cdot q \rangle = 0$. Compacta $P(q)$ are assumed to be uniformly bounded. In the terminology of [1] the action $F^{(2)}[t_*, t^*)$ on model (1.4) is the choice of the constant control $q[t] = q_*, \ t_* \leqslant t < t^*$, and the action $F^{(1)}[t_*, t^*)$ is the choice of the weakly measurable control $p[t], \ t_* \leqslant t < t^*$. According to [3] the function $\xi(t, q)$ in (1.4) can be sought from the condition

$$\xi(t_*, q) = \limsup_{t^* \to t_* + 0} \inf_{\Pi} \sup_{y^{(\gamma)}[t^*]} \left(\frac{\langle (y^{(\gamma)}[t^*] - y^{(\gamma)}[t_*]) \cdot q \rangle}{t^* - t_*} \right) \tag{1.5}$$

When the conditions from [1] are fulfilled for transformation $A_\gamma T(\vartheta, t)$ the problem on the encounter of $y[t]$ with a specified set M can be solved by solving the analogous problem for $y^{(\gamma)}[t]$. But, in its own turn, this problem can be solved on the basis of the approximation model (1.4). Therefore, the object of this paper is to set up the problem on encounter with some set M for the motions $w[t]$ of (1.4).

2. A strategy [1] prescribes the control $p[\cdot] = \{p[t],\ \tau_i \leqslant t < \tau_{i+1}\}$ as a function of $\{\tau_i,\ w[\tau_i],\ \tau_{i+1},\ q[\tau_i]\}$ so that

$$p[\cdot] = U(\tau_i,\ w[\tau_i],\ \tau_{i+1},\ q[\tau_i]) \tag{2.1}$$

For specified $w[t_0] = w_0,\ \vartheta > t_0$ and set

$$M = [\{t,\ w\} : t_0 \leqslant t \leqslant \vartheta,\ w \in M(t)] \tag{2.2}$$

the problem is to seek a strategy P which for every motion $w[t] = w[t,\ t_0,\ w_0,\ P,\ q[\cdot]]$ of (1.4) generated by it ensures the inclusion

$$w[\tau] \in M(t) \tag{2.3}$$

for at least one $\tau \in [t_0,\ \vartheta]$ for any sequence $q[\tau_i]$ $(i = 0, 1, 2, \ldots, \tau_0 = t_0)$.

In what follows the index γ in the designation of Hilbert space Y is dropped. Space Y in the strong topology is denoted Y_s and space Y in the weak topology is denoted Y_σ. Function $\xi(t,\ q)$ is assumed continuous in $[t_0,\ \vartheta] \times Y_s$ and bounded on the weak compactum $\{\|q\| \leqslant 1\}$; it is assumed to be positive-homogeneous in q, i.e., $\xi(t,\ \alpha q) = \alpha\xi(t,\ q)$ for $\alpha \geqslant 0$. We admit the condition

$$\max_{q}\ \min_{p} \langle q^* \cdot (q\xi(t,\ q) + p)\rangle = \xi(t,\ q^*) \tag{2.4}$$
$$\|q\| = 1, \quad p \in P(q)$$

for every q^*, $\|q^*\| = 1$. By motion $w[t] = w[t,\ t_*,\ w_*,\ p_*[\cdot],\ q_*[\cdot]]$ we mean a weak solution $w[t]$ of (1.4), defined by the equality

$$\langle q \cdot w[t]\rangle = \langle q \cdot w_*\rangle + \int_{t_*}^{t} \langle q \cdot (q_*\xi(\tau,\ q_*) + p_*[\tau])\rangle\, d\tau \tag{2.5}$$

which must be valid for every q .

Suppose that some topological space $\{\zeta\}$ of parameter ζ has been chosen and that a set $\{\zeta\}_\eta \subset \{\zeta\}$ has been defined for every value $\eta \in [t_0,\ \vartheta]$. The sets $\{\zeta\}_\eta$ are assumed compact and satisfying the condition

$$\{\zeta\}_{\eta_*} = \bigcap_{\eta} \{\zeta\}_\eta, \quad \eta > \eta_* \tag{2.6}$$

The parametric aggregates of sets

$$M[\zeta] = [\{t,\ y\} : \eta \leqslant t \leqslant \vartheta,\ y \in M(t,\ \zeta)] \tag{2.7}$$

and of functions $\xi(t,\ q,\ \zeta)$, where $\zeta \in \{\zeta\}_\eta$ and $\eta \in [t_0,\ \vartheta]$, are taken as chosen. Sets $M(t,\ \zeta)$ are bounded, convex and closed and satisfy the inclusion

$$M(t,\ \zeta) \subset M(t) \tag{2.8}$$

In addition, from every admissible sequence $\{t_k\ \zeta_k\}$ $(k = 1, 2, \ldots)$ we can separate a subsequence $\{t_j,\ \zeta_j\}$ for which the sets $M(t_j,\ \zeta_j)$ converge by inclusion to set $M(t_*,\ \zeta_*)$.

We make a remark. Some sections $M(t)$ of set M can be empty. Therefore, in what follows, for those expressions in which the variable t occurs as an argument in $M(t)$ or in $M(t, \zeta)$ we could stipulate that we are dealing with only those values of t for which these sets are nonempty. We shall omit such a stipulation, but will bear it in mind. In any case we shall assume that the sets $M(\vartheta)$ and $M(\vartheta, \zeta)$ are nonempty.

For a fixed value of ζ the function $\xi(t, q, \zeta)$ is continuous in $[\eta, \vartheta] \times Y_s$ and upper-semicontinuous in $[\eta, \vartheta] \times Y_\sigma$; for a fixed q this function is lower-semicontinuous with repsect to ζ. We construct the function

$$\varepsilon^\circ(t_*, w_*, \tau, \zeta) = \max_{\|q\|=1} (\langle q \cdot w_* \rangle + \int_{t_*}^{\tau} \xi(t, q, \zeta)\, dt - \rho(q, \tau, \zeta)) \qquad (2.9)$$

for $t_* \in [\eta, \tau]$ and $\tau \in [t_*, \vartheta]$. Here $\rho(q, \tau, \zeta)$ is the support function of set $M(\tau, \zeta)$, i.e.,

$$\rho(q, \tau, \zeta) = \max_y \langle q \cdot y \rangle, \quad y \in M(\tau, \zeta) \qquad (2.10)$$

On the basis of (2.9) we construct the function

$$\varepsilon_0(t, \omega) = \min_{\{\tau, \zeta\}} \varepsilon_0(t, w, \tau, \zeta), \ \tau \in [t, \vartheta], \ \zeta \in \{\zeta\}_t \qquad (2.11)$$

The minimum in (2.11), under the assumption made, is actually achieved on a certain pair $\{\tau^\circ, \zeta^\circ\}$, as follows from the properties, discussed below, of the function $\varepsilon^\circ(t, w, \tau, \zeta)$. We consider the following halfspaces:

$$Y(t, q) = [y : \langle q \cdot y \rangle \geqslant \xi(t, q)] \qquad (2.12)$$

$$Y^*(t, q, \zeta) = [y : \langle q \cdot y \rangle \leqslant \xi(t, q, \zeta)] \qquad (2.13)$$

Condition 2. 1. For any position $\{t_*, w_*\}$ for which

$$\varepsilon_0(t_*, w_*) > 0, \quad t_* < \vartheta, \ \tau^\circ > t_* \qquad (2.14)$$

for every q^*, $\|q^*\| = 1$ we can find at least one minimizing pair $\{\tau^\circ, \zeta^\circ\}$ from (2.11), for which

$$Y(t_*, q^*) \cap (\bigcap_{q^\circ} Y^*(t_*, q^\circ, \zeta^\circ)) \neq \varnothing \qquad (2.15)$$

where the intersection is taken over the set $S(t_*, w_*, \tau^\circ, \zeta^\circ)$ of all maximizing elements q° from (2.9). The symbol $\varnothing$ denotes the empty set.

The following statement is valid.

Theorem 2. 1. Let $\varepsilon_0(t_0, w_0) = 0$. If Condition 2.1 is satisfied, a strategy U exists solving the encounter problem (2.3).

3. Let us discuss the properties of function ε°. It can be verified that under the condition $\varepsilon^\circ(t_*, w_*, \tau, \zeta) > 0$ the maximum in (2.9) is achieved on elements q° with unit norm. For fixed τ and ζ the function $\varepsilon^\circ(t, w, \tau, \zeta)$ is continuous in $[\eta, \tau] \times Y_s$ and lower-semicontinuous in $[\eta, \tau] \times Y_\sigma$. The set $S(t, w, \tau, \zeta)$ of all maximizing elements q° from (2.9) for the position $\{t, w\}$, where $\varepsilon^\circ(t, w, \tau, \zeta) > 0$, is compact in Y_s. In the region $\varepsilon^\circ(t, w, \tau, \zeta) > 0$ the sets $S(t, w, \tau, \zeta)$ are strongly upper-semicontinuous by inclusion with respect to the variation of $\{t, w\}$ in $[\eta, \tau] \times Y_s$. The proof of Theorem 2.1 uses the following fact.

Lemma 3.1. Let Condition 2.1 be satisfied and

$$\varepsilon_0 (t^*, w^*) > 0 \tag{3.1}$$

and let the inequality

$$\tau^\circ > t^* + \gamma, \quad \gamma > 0 \tag{3.2}$$

be valid for all *minimizing* τ_0 from (2.11) for position $\{t^*, w^*\}$. Then for every q^*, $\| q^* \| = 1$, and $\alpha > 0$ we can choose $\delta^* > 0$ and a control $p[t] = p^*$, $t \geqslant t^*$, which satisfies the condition

$$\langle q^* \cdot p^* \rangle \geqslant 0 \tag{3.3}$$

and is such that along the solution $w[t] = w[t, t^*, w^*, p^*, q^*]$ of equation

$$w^\cdot = q^* \xi (t, q^*) + p^* \tag{3.4}$$

the inequality

$$\varepsilon_0 (t, w[t]) \leqslant \varepsilon_0 (t^*, w^*) + \alpha (t - t_*) \tag{3.5}$$

is satisfied for all $t \in [t^*, t^* + \delta^*]$.

We fix the minimizing value $\{\tau^\circ, \zeta^\circ\}$ for the given position $\{t^*, w^*\}$. As a consequence of the inequality $\varepsilon_0 (t, w[t]) \leqslant \varepsilon^\circ (t, w[t], \tau^\circ, \zeta^\circ)$ when $t \in [t^*, t^* + \gamma]$, to prove Lemma 3.1 it is sufficient to prove the inequality

$$\varepsilon^\circ (t, w[t], \tau^\circ, \zeta^\circ) \leqslant \varepsilon^\circ (t^*, w^*, \tau^\circ, \zeta^\circ) + \alpha (t - t^*) \tag{3.6}$$

We choose $\{\tau^\circ, \zeta^\circ\}$ and p^* in (3.3) from the condition

$$(p^* + q^* \xi (t^*, q^*)) \in \bigcap_{q^*} Y^* (t^*, q^\circ, \zeta^\circ) \tag{3.7}$$

$$q^\circ \in S (t^*, w^*, \tau^\circ, \zeta^\circ)$$

in accord with condition (2.15). By the definition of ε° we have

$$\Delta\varepsilon^\circ = \varepsilon^\circ (t, w[t], \tau^\circ, \zeta^\circ) - \varepsilon^\circ (t^*, w^*, \tau^\circ, \zeta^\circ) = \langle q^\circ (t) \cdot w[t] \rangle + \tag{3.8}$$

$$\int_t^{\tau^*} \xi (\varphi, q^\circ (t), \zeta^\circ) d\varphi - \rho (q^\circ (t), \tau^\circ, \zeta^\circ) - \langle q^\circ (t^*) \cdot w^* \rangle -$$

$$\int_{t^*}^{\tau^\circ} \xi (\varphi, q^\circ (t), \zeta^\circ) d\varphi - \rho (q^\circ (t^*), \tau^\circ, \zeta^\circ)$$

where $q^\circ (t)$ is the maximizing element from (2.9) for position $\{t, w[t]\}$. By the sense of $q^\circ (t)$ we have the inequality

$$\langle q^\circ (t) \cdot w^* \rangle + \int_{t^*}^{\tau^\circ} \xi (\varphi, q^\circ (t), \zeta^\circ) d\varphi - \rho (q^\circ (t), \tau^\circ, \zeta^\circ) \leqslant \langle q^\circ (t^*) \cdot w^* \rangle + \tag{3.9}$$

$$\int_{t^*}^{\tau^\circ} \xi (\varphi, q^\circ (t^*), \zeta^\circ) d\varphi - \rho (q^\circ (t^*), \tau^\circ, \zeta^\circ)$$

The inequality

$$\Delta\varepsilon^\circ \leqslant \langle q^\circ (t) \cdot (w[t] - w^*) \rangle - \int_{t^*}^{t} \xi (\varphi, q^\circ (t), \tau^\circ, \zeta^\circ) d\varphi \tag{3.10}$$

follows from (3.8) and (3.9). The motion $w[t]$ of (3.4) is strongly continuous in t. Therefore, as a consequence of the strong semicontinuity of $S(t, w, \tau^\circ, \zeta^\circ)$ with respect to $\{t, w\}$ from $[t^*, \tau^\circ] \times Y_s$, for any $\varkappa > 0$ we can find $\delta_* > 0$ such that when $|t - t^*| \leqslant \delta_* < \gamma$ we can find, for any $q^\circ(t)$, an element $q^\circ(t^*)$ satisfying the inequality

$$\| q^\circ(t) - q^\circ(t^*) \| \leqslant \varkappa \tag{3.11}$$

But then, as a consequence of the continuity of $\xi(t, q, \zeta^\circ)$ in $[t^*, \tau^\circ] \times Y_s$, from (3.10) we obtain, according to (3.4), the following inequality:

$$\Delta\varepsilon^\circ \leqslant \langle q^\circ(t^*) \cdot (p^* + q^*\xi(t^*, q^*)) \rangle (t - t^*) - \xi(t^*, q^*, \zeta^\circ)(t - t^*) + \alpha(t - t^*) \tag{3.12}$$

under the condition $t - t^* \leqslant \delta(\alpha) = \delta^*$, where $\delta(\alpha) > 0$ is a sufficiently small number. By the choice of p^* in (3.7) and by the definition (2.13) of Y^*, from (3.12) we obtain the needed inequality (3.6) and, together with it, inequality (3.5).

4. Let us discuss the properties of function ε_0. It can be verified that function $\varepsilon_0(t, w)$ is lower semicontinuous in $[t_0, \vartheta] \times Y_\sigma$ and right-continuous with respect to t in $[t_0, \vartheta] \times Y_s$ at those positions $\{t_*, w_*\}$ where the minimizing values $\tau^\circ > t_*$. Further, for some position let $\varepsilon_0(t_*, w_*) > 0$ and let the minimizing values τ° from (2.11) satisfy the condition $\tau^\circ \geqslant t_* + \gamma_*$, $\gamma_* > 0$. Then we can find $\delta > 0$ such that for all positions $\{t, w\}$ satisfying the condition

$$t - t_* \leqslant \delta, \quad t \geqslant t_*, \quad \| w - w_* \| \leqslant \delta \tag{4.1}$$

all the minimizing values τ° will satisfy the condition

$$\tau^\circ \geqslant t + \gamma, \quad \gamma > 0 \tag{4.2}$$

The following statement is valid.

Lemma 4.1. Let condition 2.1 be satisfied and let the inequalities

$$\varepsilon_0(t_*, w_*) > 0, \quad \tau^\circ > t_* + \gamma_*, \quad \gamma_* > 0 \tag{4.3}$$

be fulfilled for a given position $\{t_*, w_*\}$. Then for every q_*, $\| q_* \| = 1$, we can choose $\delta > 0$ and a control $p_*[t] \in P(q_*)$, $t \geqslant t_*$, such that for all $t \in [t_*, t_* + \delta]$ the inequality

$$\varepsilon_0(t, w[t]) \leqslant \varepsilon_0(t_*, w_*) \tag{4.4}$$

is fulfilled for the corresponding solution $w[t] = w[t, t_*, w_*, p_*[\cdot], q_*]$ of Eq. (1.4).

Let us assign a certain value of $\alpha > 0$. We consider the sheaf of all possible motions $w[t] = w[t, t_*, w_*, p[\cdot], q_*]$, $t \geqslant t_*$ of (1.4), and we single out those motions $w[t]$, $t_* \leqslant t \leqslant \tau$, whose distance from points $\{t, w\}$ of the region

$$\varepsilon_0(t, w) \leqslant \varepsilon_0(t_*, w_*) + \alpha(t - t_*) \tag{4.5}$$

do not exceed $\alpha + \alpha(t - t_*)$ for each fixed $t \in [t_*, \tau]$. As a consequence of (4.2) and of the right-continuity of function $\varepsilon_0(t, w)$ in $[t_*, \vartheta] \times Y_s$ t we can find $\delta > 0$ and, next, choose $\alpha > 0$ so small that the condition

$$\varepsilon_0(t, w) > 0, \quad \tau^\circ \geqslant t + \gamma$$

is satisfied at all the points $\{t, w\}$ mentioned.

We can now assert that for every choice of a sufficiently small $\alpha > 0$, among the motions $w[t]$ selected, we can find at least one motion $w[t]$ defined for $t_* \leqslant t \leqslant t_* + \delta$. Let us assume to the contrary that this is not so. Let $t^* \leqslant t_* + \delta$ be the upper bound of values of τ for which motions $w[t]$, $t_* \leqslant t \leqslant \tau$, of the class delineated have been defined. At first we assume that among the motions $w[t]$ selected there are no motions $\{w[t],\ t_* \leqslant t \leqslant t^*\}$. Then we continue each motion $w[t]$, $t_* \leqslant t \leqslant \tau$, up to instant t^* by choosing $p[t] \in P(q_*)$, $\tau \leqslant t < t^*$, arbitrarily. Among these motions we consider a weakly convergent sequence $w^{(k)}[t]$, $t_* \leqslant t \leqslant t^*$, $k = 1, 2, \ldots$, for which $\tau^{(k)} \to \tau^* - 0$. The weak limit $\{w_*[t],\ t_* \leqslant t \leqslant t^*\}$ of this sequence is generated by some weakly measurable control $p_*[t] \in P(q_*)$ as a consequence of the weak compactness of all possible controls $p_*[t] \in P(q_*)$ (see [4]). But then, because of the lower semicontinuity of function $\varepsilon_0(t, w)$ in $[t_0, \vartheta] \times Y_s$ we are convinced that we can find a point w such that condition (4.5) is fulfilled and the inequality

$$\| w - w_*[t] \| \leqslant \alpha + \alpha (t - t_*)$$

is valid for every point $w_*[t]$, $t_* \leqslant t \leqslant t^*$. Thus, the motion $\{w[t]\ t_* \leqslant t \leqslant t^*\}$ constructed belongs to the class delineated. We assume that $t^* < t_* + \delta$. Let w^* be a point satisfying the conditions

$$\| w^* - w_*[t^*] \| \leqslant \alpha + \alpha (t^* - t_*)$$
$$\varepsilon_0(t^*, w^*) \leqslant \varepsilon_0(t_*, w_*) + \alpha (t^* - t_*)$$

We select the vector $q^* = (w_*[t^*] - w^*) / \| w_*[t^*] - w^* \|$. If $w_*[t^*] = w^*$, we can select any vector q^*, $\| q^* \| = 1$. By Lemma 3.1 we can choose $\delta^* > 0$ and control p^* in (3.3) such that for all $t \in [t^*, t^* + \delta^*]$ the inequality

$$\varepsilon_0(t, w^*[t]) \leqslant \varepsilon_0(t^*, w^*) + \alpha (t - t^*) \leqslant \varepsilon_0(t_*, w_*) + \alpha (t - t_*) \tag{4.6}$$

is valid for the corresponding motion $w^*[t] = w[t, t^*, w^*, p^*, q^*]$, $t \geqslant t^*$, of (3.4). If here we choose the control $p[t] = p_* \in P(q_*)$, $t \geqslant t^*$, in accord with (2.4) from the condition

$$\langle q^* \cdot (q_* \xi (t^*, q_*) + p_*) \rangle \leqslant \zeta (t^*, q^*)$$

then for the motion $w_*[t] = w[t, t^*, w_*, p_*, q_*]$ generated by it and for the motion $w^*[t]$ we can obtain the estimate

$$\| w_*[t] - w^*[t] \| \leqslant \alpha + \alpha (t - t^*) + \alpha (t^* - t_*) = \alpha + \alpha (t - t_*) \tag{4.7}$$

for all $t \in [t^*, t^* + \delta_{**}]$, where $\delta_{**} > 0$ is a sufficiently small number. Thus, as a consequence of (4.6) and (4.7) the motion $\{w_*[t],\ t_* \leqslant t \leqslant t^* + \varkappa\}$, where $\varkappa > 0$ is a sufficiently small number, falls into the delineated class of motions $w[t]$. But this contradicts the choice of t^*. The contradiction proves that with a chosen $\delta > 0$ we can find, for any sufficiently small $\alpha > 0$, a solution $\{w_*[t],\ t_* \leqslant t \leqslant t_* + \delta\}$ which satisfies the conditions needed. The weak limit of the motions $\{w^{(k)}[t],\ t_* \leqslant t \leqslant t_* + \delta\}$, $k = 1, 2, \ldots,$ constructed for some sequence $\alpha_k \to 0$, is the motion $w[t]$, $t_* \leqslant t \leqslant t_* + \delta$, satisfying condition (4.4). This proves the lemma.

The next statement derives from Lemma 4.1.

L e m m a 4. 2. Let position $\{t_*, w_*\}$ satisfy the condition

$$\varepsilon_0(t_*, w_*) \leqslant \varepsilon, \quad \varepsilon > 0, \quad t_* < \vartheta$$
$$w_* \notin M^{[\alpha]}(t, \zeta) \quad \text{for} \quad \alpha \in [0, \varepsilon], \quad \zeta \in \{\zeta_{t_*}\}$$

Then for every $\tau^* \in [t_*, \vartheta]$ and $q_*, \|q_*\| = 1$ we can find at least one control $p[t] \in P(q_*)$ such that for the corresponding motion $w[t] = w[t, t_*, w_*, p[\cdot], q_*]$ of (1.4) either the condition

$$w[\tau] \in M^{[\varepsilon]}(\tau, \zeta) \quad \text{for} \quad \zeta \in \{\zeta\}_\tau \tag{4.9}$$

is fulfilled for some value of $\tau \in [t_*, \tau^*]$ or the inequality

$$\varepsilon_0(t, w[t]) \leqslant \varepsilon \tag{4.10}$$

is fulfilled for all $t \in [t_*, \tau^*]$. Here the symbol $M^{[\alpha]}$ denotes the closed α-neighborhood of set M.

Let us consider the sheaf of all possible motions $w[t] = w[t, t_*, w_*, p[\cdot], q_*]$, $t_* \leqslant t \leqslant \tau^*$, of (4.1). We assume that condition (4.9) is not fulfilled for even one of them. Suppose then, to the contrary, that condition (4.10) too is not fulfilled for even one such motion $w[t]$. Let $t^* < \tau^*$ be the upper bound of those $\tau \geqslant t_*$ for each of which we can find a motion $w[t]$ satisfying condition (4.10) when $t_* \leqslant t \leqslant \tau$. As in the proof of Lemma 4.1 we can be convinced that then we can find a motion $w[t]$ satisfying condition (4.10) when $t_* \leqslant t \leqslant t^*$, Here $\varepsilon_0(t^*, w[t^*]) \leqslant \varepsilon$ and $\tau^\circ > t^*$ for all minimizing values τ° corresponding to position $\{t^*, w[t^*]\}$. But then according to Lemma 4.1 this motion $w[t]$ can be somewhat continued past the point $t = t^*$ with the inequality $\varepsilon_0(t, w[t]) \leqslant \varepsilon$ preserved. However, this contradicts the definition of the number $t^* < \tau^*$. This contradiction proves Lemma 4.2.

5. Theorem 2.1 is proved on the basis of the material in Sect. 4 as follows. Suppose that a partitioning $\Delta = \{\tau_i\}$ of the interval $[t_0, \vartheta]$ has been chosen. Suppose that a position $w[\tau_i]$ $(\tau_i \geqslant \tau_0 = t_0)$ has been realized, for which

$$\varepsilon_0(\tau_i, w[\tau_i]) = 0$$
$$w[\tau_i] \notin M(\tau_i, \zeta) \quad \text{for} \quad \zeta \in \{\zeta\}_{\tau_i}$$

and that the instant $\tau_{i+1} \in (\tau_i, \vartheta)$ and the control $q[t] = q[\tau_i]$, $\tau_i \leqslant t < \tau_{i+1}$, have been chosen. We specify a sequence $\{\varepsilon_k > 0\}$, $k = 1, 2, \ldots$, $\lim \varepsilon_k = 0$ as $k \to \infty$. Then by Lemma 4.2 we can construct a sequence of controls $p_k[t] \in P(q[\tau_i])$ such that for some subsequence of corresponding motions $w^{(k)}[t]$ either the condition

$$\varepsilon_0(t, w^{(k)}[t]) \leqslant \varepsilon_k$$

for all $\tau_i \leqslant t \leqslant \tau_{i+1}$ or the condition

$$w[\tau^k] \in M^{[\varepsilon_k]}(\tau^{(k)}, \zeta_k), \quad \zeta_k \in \{\zeta\}_{\tau^{(k)}}$$

for some value $\tau^{(k)} \in [\tau_i, \tau_{i+1}]$ is fulfilled. In the first case, because of the weak lower semicontinuity of function $\varepsilon_0(t, w)$ in $[t_0, \vartheta] \times Y_\sigma$ the control $p_*[t]$, generating the weak limit $w_*[t]$ for the subsequence of motions $w^{(k)}[t]$, ensures condition

$$\varepsilon_0(t, w_*[t]) = \varepsilon_0(t_*, w_*)$$

for all $t \in [\tau_i, \tau_{i+1}]$. In the second case the control $p_* [t]$, generating the weak limit $w_* [t]$ for the sequence of motions $w^{(k)} [t]$, ensures condition

$$w_* [\tau_*] \in M (\tau_*, \zeta_*), \quad \zeta_* \in \{\zeta\}_{\tau_*}$$

for $\tau_* \in [\tau_i, \tau_{i+1}]$. Here $\{\tau_*, \zeta_*\}$ is the limit pair for the sequence $\{\tau^{(k)}, \zeta_k\}$ $(k = 1,2, \ldots)$. This proves the theorem.

We cite special cases when the hypotheses of Theorem 2.1 are satisfied.

1°. When $M (t, \zeta) = M (t)$ and the maximum in (2.9) is achieved on a single element q° for every position $\{t_*, w_*\}$ where $\varepsilon^\circ (t_*, w_*, \tau, \zeta) > 0, \tau > t_*$.

2°. When M is a compactum in $[t_0, \vartheta] \times Y_\sigma$ and each set $M (t, \zeta)$ is a point $M (t, \zeta) \in M (t)$, while the intersection $\bigcap_q Y (t, q) = W (t), \; q \in Y$, is nonempty for every t. In this case we define functions $\xi (t, q, \zeta)$ by the equalities

$$\xi (t, q, \zeta) = \xi_* (t, q) = \min_y \langle q \cdot y \rangle \quad \text{for } y \in W (t)$$

The fulfillment of Condition 2.1 follows from the condition that the maximum in (2.9) again is achieved on a single element q° and the halfspace $Y (t, q^\circ)$ intersects the halfspace $Y^* (t, q^\circ, \zeta) = Y_*^* (t, q^\circ)$. Therefore every halfspace $Y (t, q)$ also intersects $Y_*^* (t, q^\circ)$.

3°. When M is a compactum in $[t_0, \vartheta] \times Y_\sigma$ and each set $M (t, \zeta)$ is a point $M (t, \zeta) \in M (t)$, function $\xi (t, q, \zeta) = - \xi (t, -q)$ and function $- \xi (t, -q)$ is concave in q; the intersection of all halfspaces $Y^* (t, q, \zeta)$, $q \in Y$, is nonempty. Under the condition that function $- \xi (t, -q)$ is concave in q each halfspace $Y (t, q)$ intersects the intersection $\bigcap_q Y^* (t, q, \zeta)$, $q \in Y$. Consequently, condition (2.15) of Condition 2.1 is satisfied.

Condition 2.1 can be developed somewhat by assuming that in it we can find at least one minimizing pair $\{\tau^\circ, \zeta^\circ\}$ from (2.11), $\beta > 0$, $\varepsilon > 0$, and a continuous function $\{\tau (t), \zeta (t)\}$, $\tau (t_*) = \tau^\circ$, $\zeta (t_*) = \zeta^\circ$, $t \in [t_*, t_* + \beta]$, such that

$$Y (t_*, q^*) \cap (\textstyle\bigcap_q Y^* (t_*, q, \tau^\circ, \zeta^\circ)) \neq \varnothing$$
$$Y (t_*, q, \tau^\circ, \zeta^\circ) = [y\colon \langle q \cdot y \rangle \leqslant \xi_* (t_*, q, \tau^\circ, \zeta^\circ)]$$
$$\xi_* (t_*, q, \tau^\circ, \zeta^\circ) = \liminf_{t \to t_*} (\xi (t_*, q, \zeta^\circ) -$$
$$(t - t_*)^{-1} [\xi (\tau^\circ, q, \zeta^\circ) (\tau (t) - \tau^\circ) + \int_{t_*}^{\tau^\circ} \xi (\varphi, q, \zeta (t)) -$$
$$\zeta (\varphi, q, \zeta^\circ) \, d\varphi + \rho (q, \tau (t), \zeta (t)) - \rho (q, \tau^\circ, \zeta^\circ)])$$

for all q satisfying the condition $\| q - q^\circ \| \leqslant \varepsilon$.

6. Let us now consider the evasion problem for a system described by the unified differential Eq. (1.4). Let the set

$$M = [\{t, y\}\colon t_0 \leqslant t \leqslant \vartheta, \; y \in M (t)] \tag{6.1}$$

be specified. The problem is to find a strategy V

$$p [\cdot] = \{p [t], \; \tau_i \leqslant t < \tau_{i+1}\} = V (\tau_i, w [\tau_i], \tau_{i+1}, q [\tau_i]) \tag{6.2}$$

which for specified $\{t_0, w_0\}$ and M ensures, for every motion $w [t] = w [t, t_0, w_0, V, q [\cdot]]$ generated bv it, the evasion

$$w [t] \notin M^\varepsilon (t) \tag{6.3}$$

for all $t \in [t_0, \vartheta]$ for some value $\varepsilon > 0$ for any sequence $q[\tau_i]$ $(i = 0, 1, 2, \ldots;\ \tau_0 = t_0)$.

Suppose that the parametric aggregates of sets

$$M[\zeta, \lambda) = [\{t, y\}: \eta \leqslant t \leqslant \vartheta,\ y \in M(t, \zeta, \lambda)] \tag{6.4}$$

and the functions $\xi(t, q, \zeta)$, $\zeta \in \{\zeta\}_\eta$, $\lambda \in \{\lambda\}_\zeta$, $\eta \in [t_0, \vartheta]$, have been chosen. We assume that the sets $M(t, \zeta, \lambda)$ are defined for all $t \in [\eta_0, \vartheta]$, $\eta_0 \in [t_0, \vartheta]$, are bounded, convex, and closed, vary continuously in the Hausdorff metric as t, ζ, and λ vary, and satisfy the conditions

$$M(t) \subset \bigcup_\lambda M(t, \zeta, \lambda), \quad \lambda \in \{\lambda\}_\zeta \tag{6.5}$$

for every ζ. We assume the sets $\{\zeta\}_\eta$ and $\{\{\lambda\}_\zeta\}_\eta$ to be compacta in the corresponding spaces of parameters ζ and λ; in addition,

$$\{\{\lambda\}_\zeta\}_{\eta_*} = \bigcap_\eta \{\{\lambda\}_\zeta\}_\eta, \quad \eta < \eta_*$$

$$\{\{\lambda\}_\zeta\}_{\eta_*} = \bigcup_\eta \{\{\lambda\}_\zeta\}_\eta, \quad \eta > \eta_*$$

For a fixed value of ζ the function $\xi(t, q, \zeta)$ is continuous in $[\eta, \vartheta] \times Y_\varepsilon$ and is upper semicontinuous in $[\eta, \vartheta] \times Y_\sigma$; for fixed q this function is continuous with respect to ζ. We construct the function

$$\varepsilon^\circ(t_*, w_*, \tau, \zeta, \lambda) = \max_{\|q\| \leqslant 1} (\langle q \cdot w_* \rangle + \int_{t_*}^{\tau} \xi(t, q, \zeta)\, dt - \rho(q, \tau, \zeta, \lambda)) \tag{6.6}$$

for $t_* \in [\eta, \tau]$, $\tau \in [t_*, \vartheta]$, when the right-hand side of (6.6) is nonnegative; otherwise $\varepsilon^\circ(t_*, w_*, \tau, \zeta, \lambda) = 0$. Here $\rho(q, \tau, \zeta, \lambda)$ is the support function of set $M(t, \zeta, \lambda)$. On the basis of (6.6) we construct the function

$$\varepsilon_0(t, w) = \min_{\{\tau, \zeta, \lambda\}} \varepsilon^\circ(t, w, \tau, \zeta, \lambda) \tag{6.7}$$

$$\tau \in [t, \vartheta], \quad \zeta \in \{\zeta\}_t, \quad \lambda \in \{\lambda\}_\zeta$$

We consider the following halfspaces: $Y(t, q)$ of (2.12) and

$$Y_*(t, q, \zeta) = [\{t, y\}: \langle y \cdot q \rangle \geqslant \xi(t, q, \zeta)] \tag{6.8}$$

Condition 6.1. For any position $\{t^*, w^*\}$ for which

$$\varepsilon_0(t^*, w^*) > \beta > 0, \qquad t^* < \vartheta \tag{6.9}$$

and for every q_*, $\|q_*\| = 1$, the condition

$$Y(t^*, q_*) \cap (\bigcap_{q^\circ} Y_*(t^*, q^\circ, \zeta^\circ)) \neq \varnothing \tag{6.10}$$

is valid, where the intersection ranges over all q° from the sets $S(t^*, w^*)$ of all maximizing elements q° of (6.6), corresponding to all the minimizing values $\{\tau^\circ, \zeta^\circ, \lambda^\circ\}$ from (6.7), corresponding to position $\{t^*, w^*\}$.

The following statement is valid.

Theorem 6.1. Let $\varepsilon_0(t_*, w_*) = \gamma > \beta$. If Condition 6.1 is fulfilled, a strategy V exists solving the evasion problem (6.3) with $\varepsilon = \gamma$.

7. Let us discuss the properties of function ε_0. Under the conditions introduced this function is continuous in $[t_0, \vartheta] \times Y_\sigma$. The sets $T(t_*, w_*)$ of minimizing triples $\{\tau^\circ, \zeta^\circ, \lambda^\circ\}$ corresponding to position $\{t_*, w_*\}$ are upper semicontinuous with respect to variation of position $\{t_*, w_*\}$ in $[t_0, \vartheta) \times Y_s$ for $\varepsilon_0(t_*, w_*) > 0$, $t \geqslant t_*$. The set $S(t_*, w_*)$ of all maximizing elements q° corresponding to position $\{t_*, w_*\}$ for all minimizing values $\{\tau^\circ, \zeta^\circ, \lambda^\circ\}$ when $\varepsilon_0(t_*, w_*) > 0$ is compact in Y_σ and these sets $S(t_*, w_*)$ are upper semicontinuous by inclusion with respect to variation of $\{t_*, w_*\}$ in $[t_0, \vartheta] \times Y_\sigma$.

Lemma 7.1. When Condition 6.1 is fulfilled we can find an element

$$p^* \in P\ (q^*) \tag{7.1}$$

satisfying the inclusion

$$(p^* + q^*\xi(t^*, q^*)) \in \bigcap_{q^\circ} Y_*(t^*, q^\circ, \zeta^\circ), \quad q^\circ \in S(t^*, w^*) \tag{7.2}$$

We assume, to the contrary, that it is impossible to find such an element p^* in (7.1) and (7.2). Then by the theorem on the separation of convex sets [5] we can find a linear functional

$$f(y) = \langle q_* \cdot y \rangle, \quad \| q_* \| = 1 \tag{7.3}$$

such that

$$\langle q_* \cdot p \rangle \geqslant \alpha + \varepsilon, \quad p \in P\ (q^*) \tag{7.4}$$

$$\langle q_* \cdot y \rangle < \alpha - \varepsilon, \quad y \in \bigcap_{q^\circ} Y_*(t^*, q^\circ, \zeta^\circ), \quad q^\circ \in S(t^*, w^*), \quad \varepsilon > 0 \tag{7.5}$$

But as a consequence of condition (7.2), relations (7.4) and (7.5) signify that the intersection $\bigcap Y_*$ being considered does not intersect the halfspace $Y(t^*, q_*)$, contradicting condition (2.4) which must be fulfilled for every choice of q_*. This contradiction proves Lemma 7.1.

The proof of Theorem 6.1 relies on the following statement.

Lemma 7.2. Let Condition 6.1 be fulfilled and let $\{t^*, w^*\}$ be a position in region (6.9). Then for every q^*, $\| q^* \| = 1$, and $\alpha > 0$ we can select an admissible control $p[t] = p^*$, $t \geqslant t^*$, and $\delta > 0$ such that the inequality

$$\varepsilon_0(t, w[t] \geqslant \varepsilon_0(t^*, w^*) - \alpha(t - t^*) \tag{7.6}$$

is fulfilled for all $t \in [t^*, t^* + \delta]$ along the corresponding solution $w[t] = w[t, t^*, p^*, q^*]$ of Eq. (1.4).

We select a constant control $p[t] = p^*$ which satisfies conditions (7.1) and (7.2). Let us estimate the quantity

$$\Delta\varepsilon_0 = \varepsilon_0(t, w[t]) - \varepsilon_0(t^*, w^*) \tag{7.7}$$

By the definition (6.7) of quantity $\varepsilon_0(t, w)$ we have

$$\Delta\varepsilon_0 = \langle q^\circ(t) \cdot w[t] \rangle + \int_t^{\tau^\circ(t)} \xi(\varphi, q^\circ(t), \zeta^\circ(t))\, d\varphi - \rho(q^\circ(t), \tau^\circ(t), \zeta^\circ(t), \lambda^\circ(t)) - \tag{7.8}$$

$$\langle q^\circ(t^*) \cdot w^* \rangle - \int_{t^*}^{\tau^\circ(t^*)} \xi(\varphi, q^\circ(t^*), \zeta^\circ(t^*))\, d\varphi - \rho(q^\circ(t^*) \tau^\circ(t^*), \zeta^\circ(t^*), \lambda^\circ(t^*))$$

where $q^\circ(t)$, $q^\circ(t^*)$, $\tau^\circ(t)$, $\tau^\circ(t^*)$, $\zeta^\circ(t)$, $\zeta^\circ(t^*)$ and $\lambda^\circ(t)$, $\lambda^\circ(t^*)$ are corresponding maximizing elements from (6.6) and minimizing elements from (6.7) associated with the positions $\{t^*, w^*\}$ and $\{t, w[t]\}$, respectively. By the definition of the minimizing triple $\{\tau^\circ, \zeta^\circ, \lambda^\circ\}$, from (7.8) follows the inequality

$$\Delta\varepsilon_0 \geqslant \langle q_*^\circ(t)\cdot w[t]\rangle + \int_t^{\tau^\circ(t)} \xi(\varphi, q_*^\circ(t), \zeta^\circ(t))\,d\varphi - \rho(q_*^\circ(t), \tau^\circ(t), \zeta(t), \lambda^\circ(t)) - \langle q_*^\circ(t)\cdot w^*\rangle - \int_{t^*}^{\tau^\circ(t)} \xi(\varphi, q_*^\circ(t), \zeta(t))\,d\varphi - \rho(q_*^\circ(t), \tau^\circ(t), \zeta^\circ(t), \lambda^\circ(t)) \tag{7.9}$$

where $q_*^\circ(t)$ is the maximizing element from (6.6) for the position $\{t^*, w^*\}$, but for the triple $\{\tau^\circ(t), \zeta^\circ(t), \lambda^\circ(t)\}$. Further, it is sufficient to consider only some right-convergent sequence $\{t_k\}$ $(k = 1, 2, \ldots)$, $\lim t_k = t^* + 0$, for which $\tau^\circ(t_k) \to \tau^\circ(t^*)$, $\zeta^\circ(t_k) \to \zeta^\circ(t^*)$, $\lambda^\circ(t_k) \to \lambda^\circ(t^*)$ and $q_*^\circ(t_k) \to q^\circ(t^*)$. Then from (7.9) we have

$$\varepsilon_0(t_k, w[t_k]) - \varepsilon_0(t_*, w_*) \geqslant \langle q^\circ(t^*)\cdot(w[t_k] - w^*)\rangle - \int_{t^*}^{t} \xi(\varphi, q^\circ(t^*), \zeta^\circ(t^*))\,d\varphi - \alpha(t_k - t^*) \tag{7.10}$$

Since $w[t]$ is a solution of Eq. (1.4) when $q = q^*$ and $p = p^*$, from (7.10) follows the inequality

$$\varepsilon_0(t_k, w[t_k]) - \varepsilon_0(t^*, w^*) \geqslant [\langle q^\circ(t^*)\cdot(p^* + q^*\cdot\xi(t^*, q^*))\rangle - \xi(t^*, q^\circ(t^*), \zeta(t^*))](t_k - t^*) - \alpha(t_k - t^*) \tag{7.11}$$

if only $t_k \in [t^*, t^* + \delta]$, where $\delta > 0$ is a sufficiently small number. The inequality proving Lemma 7.2 follows from (7.11) by condition (7.2) and by the definition (6.8) of halfspace $Y_*(t^*, q^\circ, \zeta^\circ)$.

Lemma 7.3. Let Condition 6.1 be fulfilled and let $\{t_*, w_*\}$ be the position for which $\varepsilon_0(t_*, w_*) = \gamma > \beta$, $t_* < \vartheta$. Then for every q_*, $\|q_*\| = 1$, and $\tau^* \in [t_*, \vartheta]$, $\alpha > 0$, we can choose an admissible control $p[t] = p_*[t]$, $t \geqslant t_*$, such that the inequality

$$\varepsilon_0(t, w(t)) \geqslant \varepsilon_0(t_*, w_*) - \alpha(t - t_*) \tag{7.12}$$

is fulfilled for all $t \in [t_*, \tau^*]$ along the corresponding motion $w[t] = w[t, t_*, p_*[\cdot], q_*]$ of (1.4).

Let t^* be the upper bound of the values of τ for which condition (7.12) can be fulfilled for $t_* \leqslant t \leqslant \tau$. As in Sect. 4 we can verify on the basis of the continuity of function $\varepsilon_0(t, w)$ that there exists an admissible control $p_*[t]$, $t_* \leqslant t \leqslant t^*$, which ensures inequality (7.12) for $t_* \leqslant t \leqslant t^*$. But then, if $t^* < \tau^*$, according to Lemma 7.2 we can continue this control $p[t]$ somewhat past the point t^*, so that condition (7.12) is fulfilled for $t_* \leqslant t \leqslant t^* + \delta$, $\delta > 0$. But this contradicts the choice of number t^*, and this proves Lemma 7.3.

8. Theorem 6.1 is proved on the basis of the preceding material as follows. Suppose that a partitioning $\Delta = \{\tau_i\}$ of the interval $[t_0, \vartheta]$ has been chosen. Let a position $w[\tau_i]$, $(\tau_i \geqslant \tau_0 = t_0)$ be realized for which

$$\varepsilon_0(\tau_i, w[\tau_i]) = \gamma > \beta$$

and let an instant $\tau_{i+1} \in (\tau_i, \vartheta)$ and a control $q[t] = q[\tau_i]$, $\tau_i \leqslant t < \tau_{i+1}$, be chosen. We specify a sequence $\{\varepsilon_k > 0\}$ $(k = 1, 2, \ldots)$, $\lim \varepsilon_k = 0$ as $k \to \infty$. Then in accord with Lemma 7.3 we can construct a sequence of controls $p^k[t]$ such that condition

$$\varepsilon_0(t, w^{(k)}[t]) \geqslant \varepsilon_0(t_*, w_*) - \varepsilon_k$$

is fulfilled for all $\tau_i \leqslant t \leqslant \tau_{i+1}$ for some subsequence of corresponding motions $w^{(k)}[t]$. Because of the continuity of function $\varepsilon_0(t, w)$ in $[t_0, \vartheta] \times Y_\sigma$ the control $p_*[t]$ generating the weak limit $w_*[t]$ for the subsequence of motions $w^{(k)}[t]$ ensures the condition

$$\varepsilon_0(t, w_*[t]) \geqslant \varepsilon_0(t_*, w_*) = \gamma$$

for all $t \in [\tau_i, \tau_{i+1}]$. This proves Theorem 6.1.

Particular cases, when the hypotheses of Theorem 6.1 are fulfilled, are: 1) when section $M(t)$ varies continuously as t varies and each set $M(t, \zeta, \lambda)$ is a continuous curve $m(t, \zeta, \lambda) \in M(t)$ when $\eta \leqslant t \leqslant \vartheta$; for each value of t and q

$$Y(t, q) \cap W(t) \neq \varnothing$$

where the $W(t)$ are closed convex sets, and $\xi(t, q, \zeta) = \min \langle y \cdot q \rangle$ for $y \in W(t)$; 2) when under the same assumptions on sets $M(t)$ and $M(t, \zeta, \lambda)$ for each value

$$\bigcap_q Y_*(t, q, \zeta) \neq \varnothing, \quad q \in Y$$

and $\xi(t, q, \zeta) = -\xi(t, -q)$ and function $-\xi(t, -q)$ is concave in q.

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